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THE CONVEXITY OF THE REACHABLE SET FOR A BILINEAR CONTROLLABLE SYSTEM†

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The reachable of a bilinear controllable system, in which the range of allowed control values is a convex polyhedron, is examined. Sufficient conditions of convexity of the reachable set, which enable the maximum principle to be used in the standard form are found. © 2003 Elsevier Ltd. All rights reserved.

The geometry of the reachable set of non-linear controllable systems has still not been sufficiently investigated. The analysis of the reachable set of controllable systems usually confines itself to determining the sufficient conditions for its compactness [1]; in [2] the reachable set of controllable systems was analysed only during short periods of time; in [3] the sufficient condition of the so-called Λ convexity of the reachable set of controllable systems is founded; while some problems linked to the geometry of the reachable set of bilinear systems were considered in [4, 5].

Below we establish sufficient conditions for which the reachable set of a bilinear controllable system is convex. These conditions can be applied directly, unlike the results previously obtained, enabling us to use the maximum principle in standard form to analyse bilinear controllable systems effectively.

We will consider the problem of the optimum speed of response with fixed ends for a smooth controllable system

$$\dot{x} = f(x, u), \quad x \in M, \quad u(\cdot) \in \mathfrak{D} \tag{1}$$

where M is a smooth C^∞ manifold regularly embedded in $\mathbb{R}^n$, U is a convex compact polyhedron in $\mathbb{R}^m$, and $\mathfrak{D}$ is the range of allowed controls consisting of all restricted measurable functions of time t taking values in U.

Let $F(u)$ be the point of phase space to which the controllable system (1), transfer from the initial point x_0 in the time T under the influence of the allowed control $u(\cdot) \in \mathfrak{D}$. We will call the set

$$F(\mathfrak{D}) = \{F(u), u(\cdot) \in \mathfrak{D}\}$$

the reachable set $F(\mathfrak{D})$ of the controllable system (1) from the point x_0 in time T.

We will call

$$\frac{\partial F(u)}{\partial u}\eta = \lim_{\varepsilon \to 0} \frac{F(u + \varepsilon\eta) - F(u)}{\varepsilon}$$

the derivative $F(u)$ with respect to the direction $\eta(\cdot) \in \mathfrak{D}$.

The mapping

$$f : N \to M$$

of the smooth (finite-dimensional or infinite-dimensional) manifold N onto the finite dimensional manifold M is called the *mapping of constant rank* if the rank of the differential (tangential mapping)

$$\frac{\partial f(x)}{\partial x} : T_x N \to T_{f(x)} M$$

is independent of $x \in N$.

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System (1) is called a *controllable system of a constant rank* if for any point x_0 and for any T the rank of the mapping

$$u(\cdot) \mapsto F(u)$$

is independent of $u(\cdot) \in \mathfrak{D}$ (however it can depend on x_0 and T).

We will consider the bilinear controllable system

$$\dot{x} = \left(A + \sum_{i=1}^{m} B_i u_i\right)x, \quad x \in \mathbb{R}^n, \quad u(\cdot) = (u_1, \ldots, u_m) \in \mathfrak{D} \tag{2}$$

as a special case of system (1), where A and B_i ($i = 1, \ldots, m$) are square matrices of the nth order.

The sufficient conditions of constancy of the rank (relay conditions) for bilinear system (2) have the form [6]

$$[B_i, \mathrm{ad}^k AB_j]' = \sum_{\alpha=0}^{k} \sum_{\beta=1}^{m} a_{\alpha\beta}^{ijk} \mathrm{ad}^{\alpha} AB_{\beta}, \quad i, j = 1, \ldots, m, \quad k = 0, 1, \ldots \tag{3}$$

where

$$\mathrm{ad}^0 AB = B, \quad \mathrm{ad} AB = [A, B] = AB - BA$$

$$\mathrm{ad}^{k+1} AB = [A, \mathrm{ad}^k AB], \quad k = 1, 2, \ldots$$

We will assume that system (2) satisfies the relay conditions (3), i.e. it is a system of constant rank. Consequently, the plan

$$\Pi(u) = \mathrm{span}_{\mathbb{R}}\left\{\frac{\partial F(u)}{\partial u} \upsilon, \upsilon(\cdot) \in \mathfrak{D}\right\}$$

which is an image of the differential

$$\frac{\partial F(u)}{\partial u} : T_u \mathfrak{D} \to T_{F(u)} \mathbb{R}^n$$

is independent of the control $u(\cdot) \in \mathfrak{D}$, i.e. $\Pi(u) \equiv \Pi$.

Below we will denote the matrizant (the fundamental matrix) of the system

$$dX/dt = P(t)X$$

by $\Omega_0^t(P)$.

Then [6, 7]

$$\Omega_0^t(A) = e^{At}, \quad A = \mathrm{const} \tag{4}$$

$$\Omega_0^t(A + B) = \Omega_0^t(A)\Omega_0^t(\mathrm{Ad}\Omega_0^{\theta}(A)B)$$

$$\frac{\partial \Omega_0^t(A_u)}{\partial u}\eta = \Omega_0^t(A_u)\int_0^t \mathrm{Ad}\Omega_0^{\theta}(A_u)\frac{\partial A_u(\theta)}{\partial u}\eta \, d\theta \tag{5}$$

where

$$\mathrm{Ad} AB = A^{-1}BA$$

and the matrix function $A_u(t)$ depends smoothly on the parameter u.

We have

$$F(u) = \Omega_0^T(A + \mathfrak{B}u)x_0, \quad \mathfrak{B} = (B_1, \ldots, B_m), \quad u(\cdot) \in \mathfrak{D} \tag{6}$$

Theorem. The reachable set in a time T for a bilinear controllable system of constant rank (2) is convex, if a vector-function $\mu(\cdot) \in \mathfrak{D}$ and a number $\lambda \geq 0$ are founded for any vector-functions $u(\cdot), \upsilon(\cdot) \in \mathfrak{D}$ such that

$$\Omega_0^T(\mathrm{Ad}\Omega_0^\theta(e^{-\xi \mathrm{ad} A}\mathfrak{B}u)e^{-\theta \mathrm{ad} A}\mathfrak{B}(\upsilon-u)) - E = \lambda\int_0^T \mathrm{Ad}\Omega_0^\theta(e^{-\xi \mathrm{ad} A}\mathfrak{B}u)e^{-\theta \mathrm{ad} A}\mathfrak{B}(\mu-u)d\theta \tag{7}$$

where E is the identity matrix.

Proof. Let K be the non-empty set of the Banach space X. We recall (see [8, 9]) that a set of points υ is called the Clarke tangent cone $\tilde{T}K(x)$ to the set K at the point x, that for any sequence of real numbers $\{t\}$, $t_i > 0$, $t_i \to 0$ $(i \to \infty)$ and for any sequence $\{x_i\}$, $x_i \in X$, $x_i \to x$ $(i \to \infty)$ a sequence exists $\{\upsilon_i\}$, $\upsilon_i \in X$, $\upsilon_i \to \upsilon$ $(i \to \infty)$ such that

$$x_i + t_i\upsilon_i \in K$$

We will use the following convexity criterion [10] to investigate the convexity of the reachable set for bilinear system (2): the closed set K with a non-empty domain in $\mathbb{R}^n$ is convex if and only if

$$K \subset TK(x), \quad \forall x \in K$$

where $TK(x) = x + \tilde{T}K(x)$; in addition $K = \bigcap_{x \in K} TK(x)$

The Clarke tangential cone to the set $F(\mathfrak{D})$ at the point $F(u)$ can be represented [10] in the form

$$\tilde{T}F(\mathfrak{D})(F(u)) = \frac{\partial F(u)}{\partial u}\tilde{T}\mathfrak{D}(u)$$

since the mapping

$$u(\cdot) \mapsto F(u)$$

according to our assumptions, is a mapping of constant rank.

Note that since $\mathfrak{D}$ is a convex set, then

$$\mathfrak{D} \subset T\mathfrak{D}(u), \quad \forall u(\cdot) \in \mathfrak{D}$$

In particular, since

$$\upsilon \in \mathfrak{D} \subset T\mathfrak{D}(u) = u + \tilde{T}\mathfrak{D}(u)$$

then

$$\upsilon - u \in \tilde{T}\mathfrak{D}(u), \quad \forall \upsilon(\cdot) \in \mathfrak{D}$$

and, accordingly, also

$$\lambda(\upsilon - u) \in \tilde{T}\mathfrak{D}(u), \quad \forall \lambda \geq 0$$

We will introduce the following notation

$$\Theta_u^\theta = e^{-\theta \mathrm{ad} A}\mathfrak{B}u, \quad \Omega_u^T = \Omega_0^T(E_u^\theta)$$

Then, using Eqs (4) and (5) we have

$$\Omega_0^T(A + \mathfrak{B}u) = e^{TA}\Omega_0^T(\mathrm{Ad}e^{\theta A}\mathfrak{B}u) = e^{TA}\Omega_0^T(e^{-\theta \mathrm{ad} A}\mathfrak{B}u) = e^{TA}\Omega_u^T$$

and according to Eqs (6) and (7)

$$F(\upsilon) = F(u) + \Omega_0^T(A + \mathfrak{B}\upsilon)x_0 - \Omega_0^T(A + \mathfrak{B}u)x_0 = F(u) + e^{TA}(\Omega_\upsilon^T - \Omega_u^T)x_0 =$$

$$= F(u) + e^{TA}\Omega_u^T(\Omega_u^{T-1}\Omega_\upsilon^T - E)x_0 = F(u) + e^{TA}\Omega_u^T(\Omega_0^T(\mathrm{Ad}\Omega_u^\theta\Theta_{\upsilon-u}^\theta) - E)x_0 =$$

$$= F(u) + e^{TA}\Omega_u^T\lambda\int_0^T \mathrm{Ad}\Omega_u^\theta\Theta_{\mu-u}^\theta d\theta x_0 = F(u) + \frac{\partial F(u)}{\partial u}\lambda(\mu - u) \in$$

$$\in F(u) + \frac{\partial F(u)}{\partial u}\tilde{T}\mathfrak{D}(u) = F(u) + \tilde{T}F(\mathfrak{D})(u) = TF(\mathfrak{D})(u)$$

i.e.

$$F(\mathfrak{D}) \subset TF(\mathfrak{D})(u)$$

To complete the proof is remains to show that the set $F(\mathfrak{D})$ has a non-empty domain.

We will demonstrate that an affine plane containing $F(\mathfrak{D})$ exists relative to which set $F(\mathfrak{D})$ has a non-empty interior.

We will consider the plane

$$\Pi'(u) = F(u) + \Pi$$

As was shown above $F(\mathfrak{D}) \subset F(u) + \Pi$ so that $F(\upsilon) + \Pi = F(u) + \Pi$ for any $u(\cdot), \upsilon(\cdot) \in \mathfrak{D}$ i.e. the plane

$$\Pi'(u) \equiv \Pi'$$

is correctly determined.

Suppose $\dim\Pi' = k$. We will show that $F(\mathrm{int}\mathfrak{D})$ is an open subset of the plane Π'.

We will choose the functions

$$\upsilon_1(\cdot), \upsilon_2(\cdot), \ldots, \upsilon_k(\cdot) \in \mathfrak{D}$$

in such a way that the vectors

$$\frac{\partial F(u)}{\partial u}\upsilon_1, \frac{\partial F(u)}{\partial u}\upsilon_2, \ldots, \frac{\partial F(u)}{\partial u}\upsilon_k \tag{8}$$

will generate the plane Π'.

Consider the mapping (everywhere below $i, j = 1, \ldots, k$)

$$\mathscr{E} : (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \mapsto F(u + \Sigma), \quad \Sigma = \sum_{i=1}^{k} \varepsilon_i \upsilon_i, \quad \varepsilon_i \in \mathbb{R}$$

If $u(\cdot) \in \mathrm{int}\mathfrak{D}$, then the image of this mapping lies in $|\varepsilon_i|$ for sufficiently small $F(\mathrm{int}\mathfrak{D})$, since $\mathfrak{D}$ is an open set.

Further

$$\frac{\partial \mathscr{E}}{\partial \varepsilon_j} = \frac{\partial}{\partial \varepsilon_j} F(u+\Sigma) = \frac{\partial}{\partial \varepsilon_j} \Omega_0^T(A + \mathfrak{B}(u+\Sigma))x_0 =$$

$$= \Omega_0^T(A + \mathfrak{B}(u+\Sigma)) \int_0^T \mathrm{Ad}\Omega_0^\theta(A + \mathfrak{B}(u+\Sigma))\mathfrak{B}\upsilon_j(\theta)d\theta x_0$$

whence

$$\left.\frac{\partial \mathscr{E}}{\partial \varepsilon_j}\right|_{\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_k = 0} = \Omega_0^T(A + \mathfrak{B}u)\int_0^T \mathrm{Ad}\Omega_0^\theta(A + \mathfrak{B}u)\mathfrak{B}\upsilon_j(\theta)d\theta x_0 = \frac{\partial F(u)}{\partial u}\upsilon_j \tag{9}$$

Vectors (9) are linearly independent due to the linear independence of vectors (8). Consequently, the rank of mapping $\mathscr{E}$ is equal to zero in k. Hence, $F(u)$ is an internal point of the set $F(\mathrm{int}\mathfrak{D})$ by the theorem of the inverse function.

Example. Consider the bilinear controllable system

$$\dot{x}_1 = (u_1 + u_2)x_1, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u_1 x_3 \tag{10}$$

with controls

$$|u_i| \le 1, \quad i = 1, 2$$

Here

$$A = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix}, \quad B_1 = \operatorname{diag}\{1, 0, 1\}, \quad B_2 = \operatorname{diag}\{1, 0, 0\}$$

whence

$$\begin{aligned} &[B_1, B_2] = 0, \quad \operatorname{ad}AB_1 = A, \quad [B_1, \operatorname{ad}AB_1] = -\operatorname{ad}AB_1 \\ &\operatorname{ad}AB_2 = 0, \quad \operatorname{ad}^2AB_1 = 0, \quad [B_2, \operatorname{ad}AB_1] = 0 \end{aligned} \tag{11}$$

Consequently, system (10) is a system of constant rank by virtue of conditions (3).
We will introduce the notation

$$\Delta_i = v_i - u_i, \quad \Phi_i^\theta = \exp\left(\int_0^\theta u_i(\xi)d\xi\right), \quad \Psi_i^\theta = \exp\left(\int_0^\theta v_i(\xi)d\xi\right), \quad i = 1, 2$$

Using the equalities

$$e^{-\theta \operatorname{ad}A}B_1 = B_1 - \theta[A, B_1] = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & -\theta \\ 0 & 0 & 1 \end{Vmatrix}, \quad e^{-\theta \operatorname{ad}A}B_2 = B_2$$

we obtain

$$\Theta_{v-u}^\theta = e^{-\theta \operatorname{ad}A}B_1\Delta_1 + e^{-\theta \operatorname{ad}A}B_2\Delta_2 = \begin{Vmatrix} \Delta_1 + \Delta_2 & 0 & 0 \\ 0 & 0 & -\theta\Delta_1 \\ 0 & 0 & \Delta_2 \end{Vmatrix}$$

By relations (11) we have

$$\Omega_u^\theta = \begin{Vmatrix} \Phi_1^\theta\Phi_2^\theta & 0 & 0 \\ 0 & 1 & -\int_0^\theta \xi u_1(\xi)\Phi_1^\xi d\xi \\ 0 & 0 & \Phi_1^\theta \end{Vmatrix}$$

whence

$$\Omega_0^T(\operatorname{Ad}\Omega_u^\theta\Theta_{v-u}^\theta) = \begin{Vmatrix} \dfrac{\Psi_1^T\Psi_2^T}{\Phi_1^T\Phi_2^T} & 0 & 0 \\ 0 & 1 & -\int_0^T\left(\theta\Delta_1\Phi_1^\theta + \Delta_2\int_0^\theta \xi u_1(\xi)\Phi_1^\xi d\xi\right)\dfrac{\Psi_2^\theta}{\Phi_2^\theta}d\theta \\ 0 & 0 & \dfrac{\Psi_2^T}{\Phi_2^T} \end{Vmatrix}$$

Hence Eq. (7) acquires the form of a set of equalities

$$\begin{aligned}
&\int_0^T(\mu_1-u_1)d\theta = \frac{1}{\lambda}\frac{\Psi_2^T}{\Phi_2^T}\left(\frac{\Psi_1^T}{\Phi_1^T}-1\right)\\
&\int_0^T\theta(\mu_1-u_1)\Phi_1^\theta d\theta = \frac{1}{\lambda}\int_0^T\theta\frac{\Psi_2^\theta}{\Phi_2^\theta}\Delta_1\Phi_1^\theta d\theta\\
&\int_0^T(\mu_2-u_2)d\theta = \frac{1}{\lambda}\left(\frac{\Psi_2^T}{\Phi_2^T}-1\right)\\
&\int_0^T(\mu_2-u_2)\int_0^\theta\xi u_1(\xi)\Phi_1^\xi d\xi d\theta = \frac{1}{\lambda}\int_0^T\frac{\Psi_2^\theta}{\Phi_2^\theta}\Delta_2\int_0^\theta\xi u_1(\xi)\Phi_1^\xi d\xi d\theta
\end{aligned}\tag{12}$$

(if $u = \upsilon$, then Eq. (7) is trivially satisfied for $\lambda = 0$ and the arbitrary function $\mu(\cdot) \in \mathcal{D}$).

It is easy to see that if $u_1 \equiv 1$, then $\Delta_1 \le 0$ and $\Psi_1^T \le \Phi_1^T$ and if $u_1 \equiv -1$ then $\Delta_1 \ge 0$ and $\Psi_1^T \ge \Phi_1^T$. If $u_2 \equiv 1$, then $\Delta_2 \le 0$ and $\Psi_2^T \le \Phi_2^T$ and if $u_2 \equiv -1$, then $\Delta_2 \ge 0$ and $\Psi_2^T \ge \Phi_2^T$. Thus functions $\mu_1(\cdot)$, $\mu_2(\cdot)$, $|\mu_1|$, $|\mu_2| \le 1$ satisfying Eqs (12) will be found for any $u(\cdot)$, $\upsilon(\cdot)$, $|u|$, $|\upsilon| \le 1$ due to the arbitrariness of the choice of $\lambda \ge 0$.

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